

On 4-Map Graphs and 1-Planar Graphs and their Recognition Problem ^{*}

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Abstract. We establish a one-to-one correspondence between 1-planar graphs and general and hole-free 4-map graphs and show that 1-planar graphs can be recognized in polynomial time if they are crossing-augmented, fully triangulated, and maximal 1-planar, respectively, with a polynomial of degree 120, 3, and 5, respectively.

Keywords. planar graphs, 1-planar graphs, map graphs, maximality, recognition algorithms.

1 Introduction

Planarity is one of the most basic and influential concepts in graph theory. Many properties of planar graphs have been explored, including embeddings, duality, and minors. There are many linear time algorithms for their recognition as well as for the construction of straight-line grid drawings, see [37].

There were several attempts to generalize planarity to “beyond” planar graphs. Such graph allow crossings of edges with restrictions. (In other works the term near, nearly or almost planar is used). Such attempts are important, since many graphs are not planar. A prominent example is 1-planar graphs, which were introduced by Ringel [34] in an approach to color a planar graph and its dual. A graph is *1-planar* if it can be drawn in the plane such that each edge is crossed at most once. These graphs have found recent interest, in particular in graph drawing, as presented by Liotta [32]. Special cases are IC-planar and outer 1-planar graphs. A graph is *IC-planar* [14, 30, 41] if it has an embedding with at most one crossing per edge and in which each vertex is incident to at most one crossing edge. If a graph can be embedded in the plane with all vertices in the outer face and at most one crossing per edge, then it is *outer 1-planar* [2, 3, 20, 25, 26].

Beyond planarity may also be defined in terms of maps. A *map* \mathcal{M} is a partition of the sphere into finitely many regions. Each region is a closed disk and the interiors of two regions are disjoint. Some regions are labeled as *countries*, and the remaining regions are lakes or *holes* of \mathcal{M} . In the plane, we use the region of one country as the outer region, which is unbounded and encloses all other regions. An *adjacency* is defined by a touching of regions. There is a *strong*

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adjacency between two countries if the boundaries of their regions intersect in a segment and a *weak* adjacency if the boundaries intersect in a point. A map \mathcal{M} defines a graph G such that the countries of \mathcal{M} are in one-to-one correspondence with the vertices of G and there is an edge if and only if the respective countries are adjacent. Then G is called a *map graph* and \mathcal{M} the map of G .

Obviously, k regions meeting at a point induce K_k as a subgraph of G . If no more than k regions meet at a point, then \mathcal{M} is a k -map and G a k -map graph. Map graphs were introduced by Chen et al. [16] and further studied in [17, 18]. Chen et al. observed that ordinary planar graphs are the 2-map or 3-map graphs and characterized map graphs as half squares of bipartite planar graphs. Given a bipartite graph $B = (V, U, M)$, its half square $H^2(B)$ is a graph with vertices V and whose edges are the paths of length two in B . Chen et al. also proved that there are map graphs G such that $G - e$ is not a map graph.

In general, holes are necessary for the representation of graphs by maps, since, e.g., grids cannot be represented, otherwise. If \mathcal{M} has no holes, then it is a *hole-free* map and its map graph G is a *hole-free map graph*. A hole-free map looks like the dual of a planar graph. However, an adjacency at a point includes weak adjacency. Chen et al. remark that a map graph G is hole-free if and only if the boundary of each face of the bipartite graph B with $G = H^2(B)$ has exactly four or six edges. In [18] they established a cubic time recognition algorithm for hole-free 4-map graphs. They also observed that the 3-connected hole-free map graphs are exactly the triangulated 1-planar graphs. A triangulated 1-planar graph has a 1-planar embedding such that the boundary of each face consists of exactly three edges or edge segments (up to a crossing point). We shall extend this result and shall characterize 4-map graphs and hole-free 4-map graphs in terms of 1-planar graphs.

Given a (new) class of graphs, the recognition problem is always a challenge. In general, the complexity is in the range between linear time and \mathcal{NP} -completeness. Both extremes are reached by planar and 1-planar graphs. In general, 1-planarity is \mathcal{NP} -complete, as shown by Grigoriev and Bodlaender [24] and by Korzhik and Mohar [29], and it remains \mathcal{NP} -complete even for graphs of bounded bandwidth, pathwidth or treewidth [5]. \mathcal{NP} -completeness also holds for 3-connected 1-planar graphs with a given rotation system [4], i.e., the question whether a rotation system of a 3-connected graph is 1-planar. Also, IC-planarity is \mathcal{NP} -complete, both for a graph and a rotation system [14]. In addition, deciding whether a planar graph is sub-Hamiltonian is \mathcal{NP} -complete [40]. A graph is sub-Hamiltonian if it is a subgraph of a planar graph with a Hamilton circuit if and only if it admits a two-page book embedding [7].

Linear time algorithms for the recognition of planar graphs have attracted many researchers, as Patrignani's survey in [33] documents. Clearly, using any linear time algorithm for planarity, it can be checked whether an embedding is 1-planar, IC-planar and outer 1-planar, respectively. Independently and simultaneously, Auer et al. [2] and Hong et al. [25] developed linear time algorithms for the recognition of outer 1-planar graphs, see also [3, 26]. Recently, Brandenburg [12] showed that optimal 1-planarity can be decided in linear time. An

optimal 1-planar graph has $4n - 8$ edges. Moreover, Eades et al. [23] developed a linear time algorithm to test whether a rotation system is 1-planar. They described their algorithm for maximal 1-planar graphs, but it also goes through for crossing augmentations, which are defined in below.

Surprisingly, map graphs with holes are feasible as shown by Thorup [39]. Chen et al. [18] remark that Thorup's algorithm has a running time of about $O(n^{120})$, and that it does not imply a polynomial time recognition for k -map graphs and hole-free k -map graphs. They detail a cubic time recognition algorithm \mathcal{A} for hole-free 4-map graphs.

In this paper, we characterize 4-map graphs as crossing-augmented 1-planar graphs and hole-free 4-map graph as fully triangulated 1-planar graphs. The terms crossing-augmented and fully triangulated are defined in Section 3. Then we use the recognition algorithm of Chen et al. [18] to show that fully triangulated and maximal 1-planar graphs can be recognized in $O(n^3)$ and $O(n^5)$ time, respectively. Finally, we generalize the test for a 1-planar rotation system of Eades et al. [23] to crossing-augmented 1-planar graphs.

The paper is organized as follows. Section 2 describes basic definitions. In Section 3 we explore the relationship between 1-planar graphs and map graphs and establish our results. We conclude with some open problems in Section 4 and given an answer on conjectures of Chen et al. [17].

2 Foundations

We consider undirected graphs $G = (V, E)$ with n vertices and m edges. Unless otherwise stated, the graphs are simple and 2-connected. An *embedding* $\mathcal{E}(G)$ is a mapping of G into the plane or the sphere such that the vertices are mapped to distinct points and each edge is a Jordan arc between its endpoints. Crossings of incident edges with the same endpoint and self-intersections are excluded. An embedding defines a *rotation system* $\mathcal{R}(G)$, which is a cyclic list of incident edges or neighbors at each vertex. The embedding is *planar* if (the Jordan arcs of the) edges do not cross and *1-planar* if each edge is crossed at most once. We say that a graph is *planar* (*1-planar*) if it has a planar (1-planar, respectively) embedding, and accordingly for a rotation system. The embedding $\mathcal{E}(G)$ is a witness for planarity and 1-planarity, respectively, and it must satisfy the cyclic order at each vertex in case of a given rotation system.

A planar embedding of a graph partitions the plane (sphere) into *faces*, which are closed disks (except for the outer face) and are each specified by a cyclic sequence of edges (or the respective vertices) that forms the boundary. In 1-planar embeddings, a crossing subdivides an edge into two *edge segments*, and the planarization takes the crossing points and edge segments into account and treats them as vertices and edges, respectively.

Given a class of graphs \mathcal{G} , a graph $G \in \mathcal{G}$ is *planar-maximal*, *maximal* and *optimal*, respectively, if no further edge can be added to G without inducing a crossing with some edge of G , violating the defining property of \mathcal{G} , and violating the upper bound for the number of edges of graphs in \mathcal{G} , respectively. Hence,

a graph in \mathcal{G} is maximal if there is no supergraph in \mathcal{G} with the same set of vertices and a proper superset of edges, and it is optimal if there is no graph in \mathcal{G} of the same size and with more edges. Accordingly, an embedding $\mathcal{E}(G)$ of $G \in \mathcal{G}$ is maximal (planar-maximal), if any edge added to $\mathcal{E}(G)$ violates the defining properties of \mathcal{G} (or is crossed, respectively). Thus, a graph G is planar-maximal (maximal) if every embedding satisfying the properties of \mathcal{G} is planar-maximal (maximal, respectively). We call a graph G *plane-maximal* 1-planar if G has a planar-maximal embedding. In fact, every triangulated 1-planar graph is plane-maximal but not necessarily planar-maximal 1-planar. Note the difference between planar-maximal embeddings and graphs. As an example, consider $K_5 - e$, which is a maximal planar graph and whose planar embedding is planar-maximal 1-planar. However, the removed edge e can be added and drawn planar if a K_4 subgraph of $K_5 - e$ is drawn with a pair of crossing edges. Hence, $K_5 - e$ is plane-maximal and not planar-maximal 1-planar or IC-planar.

Clearly, the concepts maximal-planar, maximal and optimal coincide on planar graphs, and the maximum number of edges is $3n - 6$. Bodendiek et al. [9] showed that optimal 1-planar graphs have $4n - 8$ edges and that such graphs exist for $n = 8$ and all $n \geq 10$ [10]. The upper bound was rediscovered in many works. Bodendiek et al. also observed that there are maximal 1-planar graphs, which are not optimal. The gap in the number of edges of maximal 1-planar is quite large, as shown by Brandenburg et al. [15], who found sparse maximal 1-planar graphs with only $\frac{45}{17}n - \frac{84}{17}$ many edges. Similarly, optimal IC-planar graphs have at most $\frac{13}{4}n - 6$ edges [41] and there are optimal IC-planar achieving this bound. However, there are maximal IC-planar graphs with only $3n - 5$ edges. For the latter, consider graphs as displayed in Fig. 1 and note that maximal IC-planar graphs are supergraphs of maximal planar graphs. Auer et al. [3] observed that outer 1-planar graphs have at most $2.5n - 4$ edges, whereas there are maximal outer 1-planar graphs with $\frac{11}{5}n - \frac{18}{5}$ many edges, and both bounds are tight.

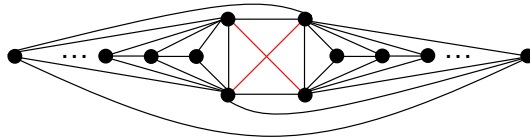


Fig. 1. Sparse maximal IC-planar graphs with $3n - 5$ edges

The complete graph on four vertices K_4 and its embedding plays a crucial role. It can be drawn planar as a *tetrahedron* and with a pair of crossing edges as a *kite*, see Fig. 2. In fact, there are four embeddings as a kite: fix a and flip

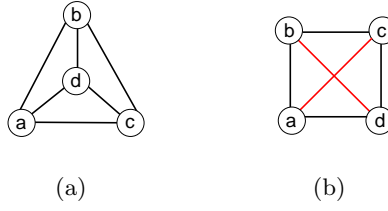


Fig. 2. Drawings of K_4 (a) planar as a tetrahedron and (b) with a crossing as a kite.

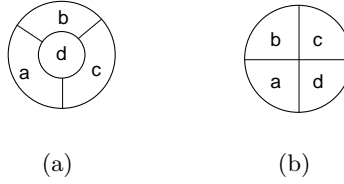


Fig. 3. A map of K_4 (a) as a rice-ball and (b) as a pizza .

b, c and c, d [31]. In the terminology of Chen et al. [17, 18] on map graphs, a tetrahedron corresponds to a rice-ball and a kite to a pizza, see Fig. 3.

We summarize the bounds on the complexity of subclasses of 1-planar graphs in Table 1.

	1-planar	IC-planar	outer 1-planar
graph	\mathcal{NP} -complete [24, 29]	\mathcal{NP} -complete [14]	$O(n)$ [3, 26]
rotation system	\mathcal{NP} -complete [4]	\mathcal{NP} -complete [14]	$O(n)$
crossing-augmented	$O(n^{120})$ [18, 39]	?	$O(n)$
fully triangulated	$O(n^3)$?	$O(n)$
plane-maximal	?	?	$O(n)$
planar-maximal	?	?	$O(n)$ [3]
maximal	$O(n^5)$?	$O(n)$ [3]
optimal	$O(n)$ [12]	?	$O(n)$

Table 1. The recognition complexity of 1-planar graphs

3 Polynomial time solvable instances

In this section we characterize 4-map and hole-free 4-map graphs in terms of 1-planar graphs and show that maximal-planar and maximal 1-planarity can be recognized in $O(n^3)$ and $O(n^5)$ time, respectively.

It was first observed by Ringel [34] and rediscovered many times that a pair of crossing edges in a 1-planar embedding can be augmented to form a kite. This augmentation seems to make the difference between tractable and intractable instances of 1-planar graphs. However, augmentation needs an embedding.

Definition 1. A 1-planar embedding $\mathcal{E}(G)$ of a 1-planar graph G is crossing-augmented if for every pair of crossing edges $\{a, b\}$ and $\{c, d\}$ in $\mathcal{E}(G)$ there are the edges $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{d, a\}$ in G .

Obviously, planar-maximal and maximal implies crossing-augmented. More importantly, we can improve upon the observation of Chen et al. [18] that the triangulated 1-planar graphs are exactly the 3-connected hole-free 4-map graphs. Triangulated means that the boundary of each face in an embedding consists of exactly three edges or edge segments (up to a crossing point), and it enforces 3-connectivity.

Theorem 1. A 1-planar graph is crossing-augmented if and only if it is a 4-map graph (with holes).

Proof. A graph $G = (V, E)$ is a 4-map graph if and only if $G = H^2(B)$ for a planar bipartite graph $B = (V, U, F)$ [17] with vertices of degree two and four in U . Construct a 1-planar embedding of G from the embedding of B by contracting the edges incident to degree-2 vertices of U and replacing each vertex of degree four of U by a kite. Then G is crossing-augmentation. Conversely, an embedded planar bipartite graph B is obtained from $\mathcal{E}(G)$ by subdividing each planar edge and replacing each crossing point by a degree-4 vertex of U such that $G = H^2(B)$. \square

The result of Thorup [39] and the remark of Chen et al. [18] is used for our first recognition problem on 1-planar graphs.

Corollary 1. For a graph G , it takes polynomial time (of degree about 120) to test whether G is crossing-augmented 1-planar.

When considering maps as dual graphs, one must avoid vertices of degree greater than four in the dual graphs if 4-maps and 4-map graphs are taken into account. Hence, the faces of an embedded graph should be triangles or look like kites. However, this presupposition is not granted if the given graph is not 3-connected. If there is a separation pair $\{u, v\}$ and $G - \{u, v\}$ decomposes into components G_1, \dots, G_k for some $k > 1$, then G is 1-planar if and only if each of the graphs $G_i + e$ is 1-planar with $e = (u, v)$ as a planar edge, as noted by Chen et al. [18]. Similarly, Brandenburg [11] has introduced copies of the edge (u, v) to separate the components at a separation pair. This idea must be retained.

Definition 2. A 1-planar embedding $\mathcal{E}(G)$ is fully triangulated if the separated embedding $\mathcal{E}_s(G)$ is triangulated, i.e., the boundary of each face consists of three edges or edge segments up to a crossing point. $\mathcal{E}_s(G)$ is obtained from $\mathcal{E}(G)$ by adding a copy of the edge (u, v) to separate components at a separation pair $\{u, v\}$. Accordingly, a graph is fully triangulated 1-planar if it admits such an embedding.

Chen et al. [18] noted that a graph G is a 4-map graph if and only if G is a triangulated 1-planar graph, provided G is 3-connected. We generalize this result to 2-connected graphs. Note that maps as well as planar dual graphs enforce 2-connectivity.

Theorem 2. A 1-planar graph is fully triangulated if and only if it is a hole-free 4-map graph.

Proof. Let $\mathcal{E}_s(G)$ be a weakly triangulated embedding. Then remove one edge from each pair of crossing edges. The resulting graph H is triangulated including copies of the edge between separation pairs. The hole-free 4-map is obtained from an embedding of the dual H^* by contracting the edge between the vertices of triangular faces, which were obtained by a removal of a crossing edge. Each contraction merges the end-vertices and results in a 4-point. The weak adjacency returns the formerly removed crossing edge. Note that H^* may have multiple edges.

Conversely, if \mathcal{M} is a hole-free 4-map of G , then take \mathcal{M} as a planar dual H^* and construct the planar primal graph H . H has multiple edges between vertices u and v if and only if the boundary between two regions is not a simple curve or path if and only if $\{u, v\}$ is a separation pair of G . At each 4-point of \mathcal{M} with countries a, b, c, d in this order, H includes the edges $(a, b), (b, c), (c, d), (d, a)$. Add the edges (a, c) and (b, d) to obtain a graph G' such that there is a kite with vertices a, b, c, d in the embedding $\mathcal{E}(G')$, which is obtained from the embedding of H . Here, multiple copies of edges are taken into account. Finally, remove all but one copy of each multi-edge from G' such that the resulting graph is simple. This graph is G . \square

Crossing-augmentation and triangulation enforce distinctions between 1-planar and map graphs. Chen et al. [17] have shown that the removal of an edge destroys map graphs. Their example can be used to show that map graphs are not closed under subdivision. On the other hand, 1-planar graphs are closed under taking subgraphs and subdivisions. In fact, every graph can be obtained from a 1-planar graph by subdivisions. Hence, neither map graphs nor 1-planar graphs can be characterized by minors.

Using the cubic time recognition algorithm of Chen et al. [18] and the SPQR-decomposition for the detection of all separation pairs [21], we immediately obtain:

Corollary 2. For a graph G , it takes $\mathcal{O}(n^3)$ time to test whether G is fully triangulated 1-planar.

Theorem 3. *For a graph G , it takes $\mathcal{O}(n^5)$ time to test whether G is maximal 1-planar.*

Proof. Clearly, a graph G is maximal 1-planar if G is maximal 1-planar and $G + e$ is not for any new edge e added to G , and each of the $\mathcal{O}(n^2)$ tests takes $\mathcal{O}(n^3)$ time. \square

We would like to establish tractability also for plane-maximal and planar-maximal 1-planar graphs. The obstacle is the variety of 1-planar embeddings. There are even optimal 1-planar graphs with different embeddings, see [35, 36]. Algorithm \mathcal{A} of Chen et al. [18] embeds a K_4 as a kite (correct pizza), whenever possible, and then “makes progress” by removing one crossing edge. However, there are places, such as a so-called separating edge, where \mathcal{A} has a choice. If \mathcal{A} computes a planar-maximal embedding of a graph G , then there may be another embedding such that a planar edge can be added. Conversely, if the computed embedding is not planar-maximal, there may be a planar-maximal one. However, we can only reduce the general case to the 3-connected case.

Definition 3. *A 1-planar embedding $\mathcal{E}(G)$ with a planar edge (a, b) in the outer face is called open if after the removal of (a, b) there is a vertex v of G in the outer face. v is called open vertex. Otherwise, G is called closed. A 1-planar graph with a distinguished edge (a, b) is open if it has an open embedding.*

A 1-planar graph is closed if its embedding is a W-configuration of Thomassen [38], see Fig. 4(a). W-configurations do not allow straight-line 1-planar drawings, as noted in [38] and [27]. An embedding is open at one or two sides. In the first case it is a B-configuration [38], see 4(b) and it has a planar interface if it is two-sided open.

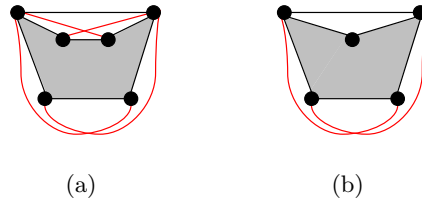


Fig. 4. A (a) W-configuration and a (b) B-configuration. There may be subgraphs in the outer face and in the shaded area

In a map, the boundary between two regions u and v is not a simple curve and looks like a chain of pearls, see Fig. 5. Each pearl represents a component G_i of $G - \{u, v\}$ at a separation pair $\{u, v\}$ and has a left and a right contact point.

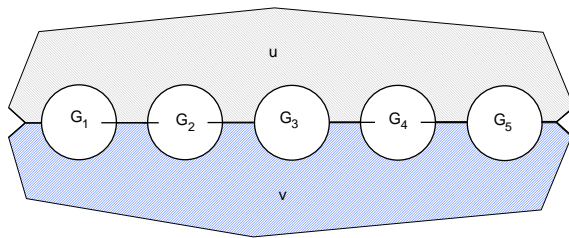


Fig. 5. A map of a separation pair $\{u, v\}$ with a chain of pearls. G_1 is open with a B-configuration, G_2 and G_4 are closed, and G_3 and G_5 are open and planar.

Now, G_i is closed if and only if both contact points are 4-points. If both contact points are 3-points, then G_i is open and planar, and it is a B-configuration, if one contact point is a 4-point.

There are many embeddings of the components at a separation pair. Each component can be flipped and the components can arbitrarily be permuted. This corresponds to a flip of the pearls and their permutation in a map.

Lemma 1. *There is a linear-time reduction from the problem of deciding whether a given 1-planar graph is planar-maximal and plane-maximal, respectively, to the special case where the graph is 3-connected.*

Proof. Clearly, a graph G is (plane or planar-maximal) 1-planar if and only if at every separation pair $\{u, v\}$ the components $G_i + e$ are (plane or planar-maximal) 1-planar, respectively, where $G - \{u, v\}$ is decomposed into components G_1, \dots, G_k for some $k > 1$ and the edge $e = (u, v)$ is planar. Each $G_i + e$ is a subgraph of G and thus remains 1-planar.

Suppose that each $G_i + e$ is planar-maximal 1-planar. Then G is planar-maximal if and only if at most one $G_i + \{u, v\}$ is open. Similarly, if each $G_i + e$ is plane-maximal 1-planar, then G is plane-maximal if and only if the number of two-sided open components does not exceed the number of components with a closed embedding. Then the components can be arranged such that two open vertices do not appear in the same face.

These properties are checked in linear time along the SPQR decomposition tree, in which the input graph is recursively decomposed at its separation pairs and at an edge if the component is 3-connected, see [21]. \square

The parallel results for IC-planar graphs are not yet clear. If algorithm \mathcal{A} finds a 1-planar embedding which is not IC, then there may be another IC-planar embedding.

For outer 1-planar graphs, Auer et al. [3] showed that the recognition of maximal-planar, maximal and outer 1-planar graphs, respectively, can be solved in linear time. They use the decomposition of a graph into its 2-connected components and retrieve planar-maximality and maximality directly from the structure of the SPQR-tree [21]. For optimal outer 1-planarity one can either check that

the given graph is outer 1-planar and has $2.5n - 4$ edges or that the SPQR-tree is composed of kites. Similarly, properties like plane-maximal, crossing-augmented and fully triangulated can directly be recognized at the SPQR-tree.

Theorem 4. *For a graph G , the following problems can be solved in linear time.*

1. *Is G outer 1-planar [3, 26]?*
2. *Is G crossing-augmented outer 1-planar?*
3. *Is G weakly triangulated outer 1-planar?*
4. *Is G plane-maximal ?*
5. *Is G planar-maximal [3]?*
6. *Is G maximal outer 1-planar [3]?*
7. *Is G optimal outer 1-planar?*

Finally, we can improve upon a result of Eades et al. [23] on the recognition of 1-planar rotation systems. The algorithm of Eades et al. considers walks around a face and finds a simple cycle if the face is planar and traverses the crossing edges twice in opposite directions before the walk revisits an edge in the same direction if there is a kite. Simply speaking, it uses the crossing edges as a bridge.

Corollary 3. *There is a linear time algorithm to test whether a rotation system is 1-planar if the underlying embedding is crossing-augmented.*

4 Conclusion and Open Problems

We showed that 1-planarity can be tested in polynomial time if the graphs are crossing-augmented, planar-maximal, maximal and optimal, respectively.

(i) Do similar results also hold for IC-planarity?

There are many other classes of beyond planar graphs, such as fan-planar [6, 8], bar 1-visibility [19] and bar (1,j)-visibility graphs [13], right angle crossing graphs (RAC) [22], quasi-planar graphs [1], and rectangle visibility graphs [28].

(ii) It is unknown whether planar-maximality, maximality and optimality in these classes can be recognized in polynomial time. For outer-fan planar graphs, maximality can be tested in linear time [6].

In general, 1-planar embeddings are not unique. However, it seems that such embeddings are weakly equivalent if the graphs are (planar-) maximal or optimal. Two embeddings $\mathcal{E}_1(G)$ and $\mathcal{E}_2(G)$ of a graph G are weakly equivalent if there is a graph automorphism $\sigma : G \rightarrow G$ such that $\mathcal{E}_1(G)$ is (topologically) equivalent to $\mathcal{E}_2(\sigma(G))$. Schumacher [35] and Suzuki [36] proved that optimal 1-planar graphs are weakly equivalent.

(iii) Do maximal (planar-maximal) 1-planar and IC-planar graphs, respectively, have a unique embedding up to weak isomorphism?

Chen et al. [17] address a generalization of maps and allow a region u to include another region. They conjecture that the recognition problem for this generalization remains polynomially solvable, which clearly holds true, since the enclosing region is an articulation vertex of the map graph. Another generalization of Chen et al. is unclear. The relation between two regions shall be left

unspecified. The regions may touch or not. If regions may touch, but the respective vertices in the map graph are not connected by an edge, then the resulting map graphs are subgraphs of 3-connected 1-planar graphs. For such graphs, the conjecture of Chen et al. holds true, since the recognition problem for 1-planar graphs is \mathcal{NP} -complete [4]. Note that there are 1-planar graphs, such as the sparse maximal 1-planar graphs of Brandenburg et al. [15], which are not a subgraph of a 3-connected 1-planar graph.

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